

Wave Equations for Classical Two-Component Proca Fields in Curved Spacetimes with Torsionless Affinities

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Abstract

The world formulation of the full theory of classical Proca fields in generally relativistic spacetimes is reviewed. Subsequently the entire set of field equations is transcribed in a straightforward way into the framework of one of the Infeld-van der Waerden formalisms. Some well-known calculational techniques are then utilized for deriving the wave equations that control the propagation of the fields allowed for. It appears that no interaction couplings between such fields and electromagnetic curvatures are ultimately carried by the wave equations at issue. What results is, in effect, that the only interactions which occur in the theoretical context under consideration involve strictly Proca fields and wave functions for gravitons.

1 Introduction

Traditionally, the Infeld-van der Waerden $\gamma\varepsilon$ -formalisms [1] constitute the classical two-component spinor framework for general relativity. The construction of these formalisms was primarily aimed at exhibiting an elementary description of the dynamics of Dirac fields in generally relativistic spacetimes. Such formalisms had been designed originally much earlier than the achievement of the definitive conditions for a curved space to admit spinor structures locally [2-4]. The legitimacy of the procedures for building up them relies crucially upon the existence of sets of Hermitian connecting objects at non-singular spacetime

points. Their affine prescriptions were formally shaped upon the ones that occur in the realm of general relativity. Thus, the generalized Weyl gauge group [5] is taken to operate on spin spaces set up locally in a way that does not depend at all upon the action of manifold mapping groups. Loosely speaking, all curvature spinors arise from the decomposition of mixed world-spin quantities that result from the action of torsionless covariant derivative commutators on arbitrary spin vectors [1, 6]. The $\gamma\varepsilon$ -formalisms were extensively utilized over the years by many authors in several different ways [6-29], noticeably enough, to reconstruct some classical generally relativistic structures and to transcribe classification schemes for world curvature tensors. Notwithstanding the fact that the construction of curvature spinors is implicitly carried by the formalisms, the spin curvatures that occur in the classification schemes and some of the spinor structures mentioned above were obtained in an artificial way by carrying out straightforward spinor translations of Riemann and Weyl tensors. A fairly complete algebraic description of the affine and curvature structures tied in with the formalisms is supplied in Refs. [30-32].

The most striking physical feature of the $\gamma\varepsilon$ -framework lies over the result that any curvature spinors are expressed as sums of purely gravitational and electromagnetic contributions which produce, in an inextricably geometric way, the occurrence of wave functions for gravitons and photons of both handednesses. The gravitational contributions for the ε -formalism were utilized in Refs. [18, 19] to support a spinor translation of Einstein's equations. It had been established a little earlier [20] that any of them should show up as a spinor pair which must be associated to the irreducible decomposition of a Riemann tensor. Any gravitational wave functions for either formalism are defined as totally symmetric curvature pieces that occur in spinor decompositions of Weyl tensors [18]. On the other hand, each electromagnetic curvature contribution emerges as a pair of suitably contracted pieces which enter the spinor representation of a locally defined Maxwell bivector [30]. The work of Ref. [4] gives a rough description of the propagation of gravitons for the ε -formalism together with a derivation of the patterns for their interactions with external electromagnetic fields. In Refs. [30, 32], the full $\gamma\varepsilon$ -description of the propagation of spin curvatures in vacuum is brought out. It thus appears that the couplings between gravitons and photons are strictly borne in both formalisms by the wave equations that govern the electromagnetic propagation. The propagation of gravitons in the presence of arbitrary sources is described in Ref. [33] where a somewhat important condition on the first covariant derivative of energy-momentum tensors is deduced. A specialization of this description for the particular case of sources coming from electromagnetic curvatures, is given in Ref. [26]. The work of Ref. [27] touches upon an interesting situation concerning the occurrence of geometric sources in the field equations for Infeld-van der Waerden photons. In Ref. [30], it was suggested for the first time that a description of some of the physical properties of the cosmic microwave background could be achieved by looking at the propagation in Friedmann-like conformally flat spacetimes of electromagnetic curvatures. A notable class of conformally flat spacetimes which admit decomposable Christoffel connexions, was considered in Ref. [21]

in conjunction with a derivation of the corresponding spin-affine and curvature configurations for the γ -formalism. It was shown thereabout that the whole derivation can actually be implemented only if a specific constancy property is imposed on one of the spin densities borne by the expression for a characteristic γ -metric function. Explicit expressions for the gravitational spinors of those spacetimes were then derived. A detailed description of the interaction couplings that take place in the formulation of Dirac's theory in curved spacetimes has likewise been given [22]. This latter work has really made up the original description of Dirac fields as given by Infeld and van der Waerden.

In the present work, we exhibit the formulation of the theory of classical Proca fields within the framework of the γ -formalism. The theory of spin densities and the gauge transformations inherently borne by the ε -formalism [30, 31] will not be exhibited by this point since the ε -counterparts of our key developments do not bring forth any further formal insight. The spinor field equations are obtained out of transcribing directly the statements that make out the world version of the theory. Some well-known calculational techniques are then utilized for deriving the wave equations that control the propagation of the fields taken into consideration. Indeed, these techniques are just the same as the ones employed in Refs. [23, 30] for obtaining the typical wave equations of the entire $\gamma\varepsilon$ -framework. Hence, no interaction couplings between Proca fields and electromagnetic curvatures are ultimately carried by the resulting wave equations. What comes about is, in effect, that the only interactions which occur in the theoretical context being considered involve strictly Proca fields and wave functions for gravitons. One of our motivations for elaborating upon the situation entertained herein is related to the absence from the literature of any systematic two-component description of the propagation of massive spinning bosons in generally relativistic spacetimes. In our view, it might be worthwhile to work out such a massive case towards exhibiting the patterns of the couplings that should arise in the pertinent context. It is from this fact that the main physical aim of our paper stems.

We will adopt the notation adhered to in Ref. [30] except that spacetime components will now be labelled by lower-case Greek letters. Kernel letters for world and spin quantities will broadly appear as Greek and Latin letters. In particular, we denote as x^μ some local coordinates on a spacetime \mathfrak{M} equipped with a torsionless covariant derivative operator ∇_μ . A world metric tensor $g_{\mu\nu}$ on \mathfrak{M} presumably bears the local signature $(+ - - -)$. We thus require $g_{\mu\nu}$ to fulfill at the outset the metric compatibility condition of general relativity

$$\nabla_\mu g_{\lambda\sigma} = 0,$$

which means that we shall allow for the (unique) Levi-Civita connection associated to ∇_μ . The partial derivative operator for x^μ is denoted by ∂_μ , and the Riemann tensor of ∇_μ is written as $R_{\mu\nu\lambda\sigma}$. Our sign convention for the respective Ricci tensor $R_{\mu\nu}$ is the same as the one adopted in Ref. [4]. The determinant of $g_{\mu\nu}$ and the covariant alternating world density in \mathfrak{M} will especially be denoted as \mathfrak{g} and $\epsilon_{\mu\nu\lambda\sigma}$, respectively. We shall use the primed-unprimed index notation of Ref. [4] upon dealing with conjugate spinor components. World

indices all range over the four values 0, 1, 2, 3 whereas spinor indices take either the values 0, 1 or 0', 1'. We will utilize the convention according to which the effect on any index block of the actions of the symmetry and antisymmetry operators is indicated by surrounding the indices singled out with round and square brackets, respectively. A horizontal bar lying over a kernel letter will sometimes be used to denote the operation of complex conjugation. Further conventions will be explained in due course.

Our outline has been set as follows. For the sake of consistency, the world version of the Proca theory is formulated in Section 2 on the basis of the standard least-action principle for classical fields in curved spacetimes [34]. Section 3 brings out the overall system of spinor field equations. In Section 4, we carry out the derivation of our wave equations. Some remarks on our work are made in Section 5. The calculational techniques referred to above shall be taken for granted from the beginning.

2 World theory

The least-action principle for the Proca theory in \mathfrak{M} is written as

$$\delta S = \delta \int_{\Omega} \mathcal{L} \sqrt{-g} d^4x = 0, \quad (1)$$

where \mathcal{L} denotes the Lagrangian density

$$\mathcal{L} = -\frac{1}{4} f^{\mu\nu} f_{\mu\nu} + \frac{1}{2} m^2 A^\mu A_\mu, \quad (2)$$

which carries the Proca bivector

$$f_{\mu\nu} = 2\partial_{[\mu} A_{\nu]} = 2\nabla_{[\mu} A_{\nu]}, \quad (3)$$

with A_μ and m being a Proca potential and the mass of $f_{\mu\nu}$. Usually, the variation δ bears linearity and obeys the Leibniz rule, in addition to being defined so as to commute with partial derivatives and integrations. The integral of Eq. (1) is taken over a volume Ω in \mathfrak{M} whose closure is compact, and

$$d^4x = \frac{1}{4!} \epsilon_{\mu\nu\lambda\sigma} dx^\mu \wedge dx^\nu \wedge dx^\lambda \wedge dx^\sigma \quad (4)$$

defines an elementary volume density in Ω , with the symbol “ \wedge ” thus denoting the wedge product.

With the help of Eqs. (2) and (3), we can rewrite the statement (1) as

$$\delta S = \int_{\Omega} (-f^{\mu\nu} \partial_\mu \delta A_\nu + m^2 A^\nu \delta A_\nu) \sqrt{-g} d^4x = 0, \quad (5)$$

where δA_ν is taken as an arbitrary covariant quantity in Ω that vanishes on the

boundary $\partial\Omega$ of Ω . Hence, performing an integration by parts in (5), yields

$$\begin{aligned} & \int_{\Omega} \left[\frac{1}{\sqrt{-\mathfrak{g}}} \partial_{\mu} (\sqrt{-\mathfrak{g}} f^{\mu\nu}) + m^2 A^{\nu} \right] \sqrt{-\mathfrak{g}} \delta A_{\nu} d^4x \\ & - \int_{\partial\Omega} f^{\mu\nu} \sqrt{-\mathfrak{g}} \delta A_{\nu} d^3x_{\mu} = 0, \end{aligned} \quad (6)$$

with

$$d^3x_{\mu} = \frac{1}{3!} \epsilon_{\mu\nu\lambda\sigma} dx^{\nu} \wedge dx^{\lambda} \wedge dx^{\sigma}, \quad (7)$$

whence we can write down the field equations

$$\frac{1}{\sqrt{-\mathfrak{g}}} \partial_{\mu} (\sqrt{-\mathfrak{g}} f^{\mu\nu}) + m^2 A^{\nu} = 0, \quad (8)$$

which amount to the same thing as

$$\nabla_{\mu} f^{\mu\nu} + m^2 A^{\nu} = 0. \quad (9)$$

Equations (9) constitute the first world half of Proca's theory in \mathfrak{M} . The second half comes into play as the Bianchi identity

$$\nabla_{[\mu} f_{\lambda\sigma]} = 0, \quad (10)$$

which can be reexpressed as

$$\nabla^{\mu} f_{\mu\nu}^* = 0, \quad (11)$$

where

$$f_{\mu\nu}^* = \frac{1}{2} \sqrt{-\mathfrak{g}} \epsilon_{\mu\nu\lambda\sigma} f^{\lambda\sigma} \quad (12)$$

is the dual bivector of $f_{\mu\nu}$. It follows that, by taking the covariant divergence of the left-hand side of Eq. (9), likewise implementing the commutator expansion

$$[\nabla_{\mu}, \nabla_{\nu}] f^{\mu\nu} = R_{\mu\nu\lambda}{}^{\mu} f^{\lambda\nu} + R_{\mu\nu\lambda}{}^{\nu} f^{\mu\lambda} = 2R_{\mu\nu} f^{\mu\nu} \equiv 0, \quad (13)$$

we promptly arrive at

$$\nabla_{\mu} A^{\mu} = 0. \quad (14)$$

A similar procedure involves inserting (3) into (9) to get

$$(\square + m^2) A_{\mu} - \nabla_{\lambda} \nabla_{\mu} A^{\lambda} = 0, \quad (15)$$

whence, making use of the equality

$$\nabla_{\lambda} \nabla_{\mu} A^{\lambda} = \nabla_{\mu} (\nabla_{\lambda} A^{\lambda}) + R_{\lambda\mu\sigma}{}^{\lambda} A^{\sigma}, \quad (16)$$

and calling upon (14), we end up with the wave equation

$$(\square + m^2) A_{\mu} + R_{\mu}{}^{\lambda} A_{\lambda} = 0. \quad (17)$$

3 Spinor field equations

By definition, any Proca wave functions enter formal bivector expansions like

$$\sigma_{AA'}^\mu \sigma_{BB'}^\nu f_{\mu\nu} = f_{AA'BB'} = \gamma_{A'B'} \psi_{AB} + \gamma_{AB} \psi_{A'B'}, \quad (18)$$

with

$$\psi_{AB} = \frac{1}{2} f_{ABC'}{}^{C'} = \psi_{(AB)}, \quad \psi_{A'B'} = \frac{1}{2} f_{A'B'C}{}^C = \psi_{(A'B')}, \quad (19)$$

and $(\gamma_{AB}, \gamma_{A'B'})$ being a pair of covariant metric spinors for the γ -formalism. The σ -symbols carried by Eq. (18) are some appropriate Hermitian connecting objects, which supposedly fulfill the covariant constancy requirement (see, for instance, Ref. [32])

$$\nabla_\mu \sigma_{BB'}^\lambda = 0. \quad (20)$$

Accordingly, either of ψ_{AB} and $\psi_{A'B'}$ is a massive spin-one uncharged field that represents locally the six degrees of freedom of $f_{\mu\nu}$ in \mathfrak{M} . The corresponding field-potential relationships are given by

$$\psi_{AB} = -\nabla_{(A}^{C'} A_{B)C'}, \quad \psi_{A'B'} = -\nabla_{(A'}^C A_{B')C} \quad (21)$$

and

$$\psi^{AB} = \nabla_{C'}^{(A} A^{B)C'}, \quad \psi^{A'B'} = \nabla_C^{(A'} A^{B')C}. \quad (22)$$

In passing, we point out that, in deriving Eqs. (21) and (22), it may be necessary to implement the Infeld-van der Waerden eigenvalue equations [6, 31]

$$\nabla_\mu \gamma_{BC} = i\beta_\mu \gamma_{BC}, \quad \nabla_\mu \gamma^{BC} = -i\beta_\mu \gamma^{BC}, \quad (23)$$

as well as their complex conjugates. The quantity β_μ amounts to the world vector

$$\beta_\mu = \nabla_\mu \Phi + 2\Phi_\mu, \quad (24)$$

which is invariant under the action of the Weyl gauge group [1, 6], with Φ and Φ_μ being, respectively, the polar argument of the independent component of γ_{AB} and a γ -formalism electromagnetic potential. It is useful to introduce the Maxwell bivector associated to Φ_μ . We have, in effect,

$$F_{\mu\nu} = 2\partial_{[\mu} \Phi_{\nu]} = 2\nabla_{[\mu} \Phi_{\nu]}. \quad (25)$$

The spinor decomposition of $F_{\mu\nu}$ takes up the wave functions ϕ_{AB} and $\phi_{A'B'}$ which thus supply dynamical states for Infeld-van der Waerden photons in \mathfrak{M} . These wave functions essentially constitute the electromagnetic curvature of \mathfrak{M} , thereby being deeply rooted into the geometric structure of \mathfrak{M} . It is obvious that the geometric field-potential relationships may right away be attained from (21) and (22) by making trivial replacements of kernel letters. Such relationships shall be utilized later in Section 4.

The first spinor half of Proca's theory arises here from the two-component transcription of Eq. (9) whence, by invoking (20), we obtain the field equations¹

$$\nabla^{AA'}(\gamma_{A'B'}\psi_{AB} + \text{c.c.}) + m^2 A_{BB'} = 0. \quad (26)$$

The second half now consists of the statements

$$\nabla^{AA'} f_{AA'BB'}^* = i\nabla^{AA'}(\gamma_{AB}\psi_{A'B'} - \text{c.c.}) = 0, \quad (27)$$

which effectively account for the dual expansion

$$\sigma_{AA'}^\mu \sigma_{BB'}^\nu f_{\mu\nu}^* = i(\gamma_{AB}\psi_{A'B'} - \gamma_{A'B'}\psi_{AB}). \quad (28)$$

Of course, Eq. (27) may be reset as the Hermitian configuration

$$\nabla^{AA'}(\gamma_{A'B'}\psi_{AB}) = \nabla^{AA'}(\gamma_{AB}\psi_{A'B'}). \quad (29)$$

We should also observe that the pattern of Eq. (28) oftenly emerges from the combination of (18) with the alternating expansion

$$\sqrt{-g}\epsilon_{AA'BB'CC'DD'} = i(\gamma_{AC}\gamma_{BD}\gamma_{A'D'}\gamma_{B'C'} - \gamma_{AD}\gamma_{BC}\gamma_{A'C'}\gamma_{B'D'}). \quad (30)$$

Typically, the entire Proca theory in \mathfrak{M} is written out explicitly as the field equations

$$\nabla_{B'}^A \psi_{AB} + \frac{1}{2}m^2 A_{BB'} - i\beta_{B'}^A \psi_{AB} = 0 \quad (31)$$

and

$$\nabla_A^{B'} \psi^{AB} - \frac{1}{2}m^2 A^{BB'} + i\beta_A^{B'} \psi^{AB} = 0, \quad (32)$$

together with the complex conjugates of (31) and (32). A formal simplification to it can be accomplished by utilizing Eqs. (23) along with metric prescriptions of the type

$$\nabla^{AB'} \psi_{AB} = \nabla^{AB'}(\psi_A^C \gamma_{CB}), \quad \nabla_{AB'} \psi^{AB} = \nabla_{AB'}(\gamma^{BC} \psi_C^A). \quad (33)$$

For the unprimed wave functions, for instance, we thus have the equivalent statements

$$\nabla^{AB'} \psi_A^B + \frac{1}{2}m^2 A^{BB'} = 0, \quad \nabla_{AB'} \psi_B^A - \frac{1}{2}m^2 A_{BB'} = 0. \quad (34)$$

Evidently, the symmetry borne by the wave functions makes it immaterial to order their indices.

¹The symbol "c.c." will henceforth denote an overall complex conjugate piece.

4 Wave equations

At this stage, we shall follow up the procedure which amounts to implementing the calculational techniques mentioned in Section 1 towards deriving the wave equations for the fields that occur in the statements (31)-(34). We will initially work out the procedure for ψ^{AB} and ψ_A^B . The wave equation for ψ_{AB} will then be obtained by taking into effect a valence interchange rule that had been deduced originally [30] in connection with the presentation of the general description of $\gamma\varepsilon$ -curvatures. We may certainly get the wave equations for any primed fields by taking complex conjugates. Equations (23) will be used so many times in what follows that we will no longer refer to them explicitly.

We start by operating with $\nabla_{CB'}$ on the configuration of Eq. (32). Hence, using the operator correlation

$$\nabla_{CB'}\nabla_A^{B'} = \gamma_{LC}\nabla_{B'}^L(\gamma_{MA}\nabla^{MB'}) = i\beta_{CB'}\nabla_A^{B'} + \gamma_{LC}\gamma_{MA}\nabla_{B'}^L\nabla^{MB'}, \quad (35)$$

together with the splitting

$$\nabla_{B'}^L\nabla^{MB'} = \Delta^{LM} + \frac{1}{2}\gamma^{LM}\square \quad (36)$$

and the definition²

$$\Delta^{AB} = \nabla_{C'}^{(A}\nabla^{B)C'}, \quad (37)$$

we get the contribution

$$\nabla_{CB'}\nabla_A^{B'}\psi^{AB} = i\beta_{CB'}\nabla_A^{B'}\psi^{AB} + (\Delta_{AC} - \frac{1}{2}\gamma_{AC}\square)\psi^{AB}. \quad (38)$$

The Δ -derivative of (38) reads

$$\Delta_{AC}\psi^{AB} = \frac{R}{6}\gamma_{CA}\psi^{AB} + \Psi_{AMC}{}^B\psi^{AM} - 2i\phi_{AC}\psi^{AB}, \quad (39)$$

where ϕ_{AB} stands for a wave function for Infeld-van der Waerden photons and Ψ_{ABCD} is a wave function for gravitons in \mathfrak{M} . For the β -term of (38), we have

$$i\beta_{CB'}\nabla_A^{B'}\psi^{AB} = i(\beta_{B'(A}\nabla_{C')}^{B'} - \frac{1}{2}\gamma_{AC}\beta^\mu\nabla_\mu)\psi^{AB}. \quad (40)$$

In addition, recalling the unprimed relation of (22), produces the following expansion for the differential kernel of the operated mass term coming from (32):

$$\nabla_{CB'}A^{BB'} = \gamma_{MC}(\psi^{MB} + \frac{1}{2}\gamma^{MB}\nabla_\mu A^\mu), \quad (41)$$

which, by virtue of Eq. (14), may be simplified to

$$\nabla_{CB'}A^{BB'} = \gamma_{MC}\psi^{MB}. \quad (42)$$

²The object \square equals the covariant D'Alembertian operator $\nabla^\mu\nabla_\mu$ whilst Δ^{AB} is linear and enjoys the Leibniz rule property.

We have next to allow for the contribution

$$\nabla_{CB'}(i\beta_A^{B'}\psi^{AB}) = (i\nabla_{CB'}\beta_A^{B'})\psi^{AB} + i\beta_A^{B'}\nabla_{CB'}\psi^{AB}. \quad (43)$$

It is evident that the sum of the β -term of (38) with the second term lying on the right-hand side of (43), bears skewness in the indices A and C , that is to say,

$$i\beta_{CB'}\nabla_A^{B'}\psi^{AB} + i\beta_A^{B'}\nabla_{CB'}\psi^{AB} = i\gamma_{CA}\beta_\mu\nabla^\mu\psi^{AB}. \quad (44)$$

For the other individual term of (43), we spell out the auxiliary configurations

$$i\nabla_{B'[C}\beta_A^{B']} = \frac{1}{2}\gamma_{AC}(\beta^\mu\beta_\mu - i\nabla_\mu\beta^\mu) \quad (45)$$

and

$$i\nabla_{B'(C}\beta_A^{B')} = i(\Delta_{AC}\Phi + 2\phi_{AC}), \quad (46)$$

where Φ is given by Eq. (24). The Δ -derivative of (46) vanishes identically³ because of the torsionlessness of ∇_μ . It follows that, fitting pieces together, yields

$$(\square + 2i\beta^\mu\nabla_\mu + \Theta + \frac{R}{3} + m^2)\psi^{AB} - 2\Psi^{AB}{}_{LM}\psi^{LM} = 0, \quad (47)$$

with

$$\Theta \doteq -\beta^\mu\beta_\mu + i\nabla_\mu\beta^\mu. \quad (48)$$

The entire derivation of the wave equation for the field involved in Eqs. (34) does not produce any couplings other than a gravitational one which looks like that borne by (47). Roughly speaking, the only reason for this rests upon the result that we can carry out the relevant derivation without having to call for any correlations like (35) or (45), with the valence pattern of ψ_A^B accordingly ensuring the absence of any $\phi\psi$ -interactions.⁴ For the first of Eqs. (34), say, we thus reexpress (36) as

$$\nabla_{CB'}\nabla^{AB'} = \gamma_{MC}(\Delta^{AM} - \frac{1}{2}\gamma^{AM}\square), \quad (49)$$

and let the splitting (49) act on ψ_A^B such that the relation (42) still holds. Consequently, by taking account of the derivative

$$\Delta_C^A\psi_A^B = \frac{R}{6}\psi_C^B + \Psi^{AB}{}_{CD}\psi_A^D, \quad (50)$$

while resetting the kernel for the mass term as

$$\nabla_{CB'}A^{BB'} = \psi_C^B, \quad (51)$$

³Within the $\gamma\varepsilon$ -framework, the quantity Φ is looked upon as a world scalar subject to a suitable gauge behaviour.

⁴The work of Ref. [30] describes in detail on the basis of the theory of spin densities the situation related to the eventual absence of electromagnetic contributions from Δ -derivatives.

and making some index substitutions thereafter, we obtain

$$(\square + \frac{R}{3} + m^2)\psi_A^B + 2\Psi_{AD}{}^{BC}\psi_C^D = 0. \quad (52)$$

It is worth pointing out that the Δ -derivative of (50) possesses the property

$$\Delta^{A[C}\psi_A^{B]} = 0. \quad (53)$$

The wave equation for ψ_{AB} can indeed be derived from (47) by applying to it the simultaneous interchanges

$$i\beta^\mu\nabla_\mu \leftrightarrow -i\beta^\mu\nabla_\mu, \quad \Theta \leftrightarrow \bar{\Theta}, \quad (54)$$

which come naturally from the utilization of the differential devices

$$\square\psi_{AB} = \square(\psi^{CD}\gamma_{CA}\gamma_{DB}), \quad \square(\gamma_{CA}\gamma_{DB}) = -\bar{\Upsilon}\gamma_{CA}\gamma_{DB} \quad (55)$$

and

$$2(\nabla_\mu\psi^{CD})\nabla^\mu(\gamma_{CA}\gamma_{DB}) = 4(2\beta^\mu\beta_\mu + i\beta^\mu\nabla_\mu)\psi_{AB}, \quad (56)$$

with the definition

$$\Upsilon \doteq 2(\beta^\mu\beta_\mu - \bar{\Theta}). \quad (57)$$

We thus have

$$(\square - 2i\beta^\mu\nabla_\mu + \bar{\Theta} + \frac{R}{3} + m^2)\psi_{AB} - 2\Psi_{AB}{}^{LM}\psi_{LM} = 0. \quad (58)$$

5 Concluding remarks

We saw that the coupling $2i\phi_{AC}\psi^{AB}$ occurs through the expansions (39) and (46) in the derivation that leads to the wave equation (47), but it nevertheless turns out to be cancelled when the derivation is actually carried through. If we had instead worked out the derivation procedure for ψ_{AB} , then such a $\phi\psi$ -coupling would have once again arisen at some intermediate calculational steps as can clearly be seen from the combined configurations

$$(2\Delta^{AC} - \gamma^{AC}\square)\psi_{AB} + m^2\nabla_{B'}^C A_B^{B'} = 2i\nabla_{B'}^C(\beta^{AB'}\psi_{AB}),$$

$$2\Delta^{AC}\psi_{AB} = \frac{R}{3}\psi_B^C - 2\Psi_B{}^{CMN}\psi_{MN} + 4i\phi^{AC}\psi_{AB}$$

and

$$(\nabla_{B'}^{(C}\beta^{A)B'})\psi_{AB} = \left(\Delta^{AC}\Phi + 2\phi^{AC}\right)\psi_{AB}, \quad \nabla_{B'}^{[C}\beta^{A]B'} = \frac{1}{2}\gamma^{CA}\nabla_{\mu}\beta^\mu.$$

Thus, by using the prescription

$$\nabla_{CB'}A_B^{B'} = \nabla_{CB'}(\gamma^{B'C'}A_{BC'}),$$

likewise invoking one of the relationships (21) and implementing (14), we could rearrange the kernel of the differentiated mass term of Eq. (31) as

$$\nabla_{CB'} A_B^{B'} = \psi_{BC} + i\beta_{CB'} A_B^{B'},$$

which particularly carries the potential coupling

$$i\Phi_{CB'} A_B^{B'} = i(\Phi_{B'(B} A_{C)}^{B'} - \frac{1}{2}\gamma_{BC}\Phi_\mu A^\mu).$$

In fact, the desirable covariance of Eq. (58) under the geometrically intrinsic γ -formalism gauge transformation [30]

$$\Phi_\mu \mapsto \Phi_\mu - \partial_\mu \theta,$$

is brought about when we call for the contribution

$$2i\beta^{AB'} \nabla_{CB'} \psi_{AB} = (2i\beta^\mu \nabla_\mu + \beta^\mu \beta_\mu) \psi_{BC} + im^2 \beta_{CB'} A_B^{B'},$$

which accordingly entails the cancellation of all Φ -potential couplings.

It should be obvious that both of the wave equations (47) and (58) could be readily derived from (52) by taking into account the correlations

$$(\square \psi_A^C) \gamma_{CB} = (\square - 2i\beta^\mu \nabla_\mu + \bar{\Theta}) \psi_{AB}$$

and

$$\gamma^{AC} (\square \psi_C^B) = (\square + 2i\beta^\mu \nabla_\mu + \Theta) \psi^{AB},$$

along with the eigenvalue equations

$$\square \gamma_{AB} = \Theta \gamma_{AB}, \quad \square \gamma^{AB} = \bar{\Theta} \gamma^{AB}.$$

The work we have just presented has provided us with the characteristic patterns of the γ -formalism version of the theory of classical Proca fields. We emphasize that one of the most remarkable properties of the wave equations deduced previously, is that the only interactions carried by them involve Proca fields and gravitational wave functions. Hence, Proca fields propagate in \mathfrak{M} as if Infeld-van der Waerden electromagnetic curvatures were absent. Therefore, we could say that our work has filled in the gap associated to the absence of a formal two-component description of external massive spin-one fields in general relativity. We believe that it would be of considerable interest to obtain the physically meaningful couplings involving external spinning fields, which should arise within the torsional framework of Refs. [35, 36].

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